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Clusters and droplets in the q -state Potts model†

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Abstract. A Potts correlated polychromatic percolation is studied. The clusters are made of sites corresponding to a given value of the q -state Potts variables, connected by bonds being active with probability p_B . To treat this problem an s -state Potts Hamiltonian diluted with q -state Potts variables (instead of lattice gas variables) is introduced to which the Migdal–Kadanoff renormalisation group is applied. It is found for a particular choice of $p_B = 1 - e^{-K}$ (where K is the Potts coupling constant divided by the Boltzmann factor) that these clusters, called droplets, diverge at the Potts critical point with Potts exponents.

Cluster models have often been employed to describe phase transitions (Fisher 1967, Binder 1976, Domb 1976). One major problem is the definition of a proper cluster, usually called a droplet, which is required to diverge at the critical point with the right exponents.

In site percolation problem clusters are defined as the maximal sets of nearest-neighbour particles distributed on a lattice. In standard percolation the particles are randomly distributed, while in correlated percolation they interact (see review articles by Stauffer (1979) and Essam (1980)). The case studied most is that in which the particles are correlated with ferromagnetic interactions as in a lattice gas or the Ising model (in the Ising model the clusters are made of 'down' spins). Although these Ising clusters diverge at the Ising critical point for dimensionality $d = 2$ (Coniglio *et al* 1977), this is not the case for $d = 3$ (Müller-Krumbhaar 1974).

More recently a generalisation of this problem, the site bond Ising correlated percolation, has been proposed as a model for gelation (Coniglio *et al* 1979). In this model a cluster is made of nearest-neighbour particles connected by active bonds, the particles interact according to the lattice gas Hamiltonian, the probability of a bond being active is p_B and non-active $1 - p_B$. Note that the bonds are only introduced to define the connectivity between two nearest-neighbour particles and do not affect their interacting energy and therefore the particle distribution. Coniglio and Klein (1980), using the lattice gas Potts formulation of this model, have shown that these new clusters diverge at the Ising critical point for any d provided that $p_B = 1 - \exp(-2K_I\beta)$ where K_I is the nearest-neighbour Ising interaction and $\beta = (k_B T)^{-1}$. They called these clusters Ising droplets. Very recently the site bond Ising correlated

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percolation has also been investigated by Monte Carlo methods (Stauffer 1981, Rousseng 1981) with results in agreement with the above predictions.

In the same spirit here we treat site bond Potts correlated percolation (SBPCP). Consider the usual q -state Potts Hamiltonian

$$-\beta\mathcal{H}_P = K \sum_{\langle ij \rangle} (\delta_{\nu_i \nu_j} - 1) - H \sum_i (\delta_{\nu_i, 1} - 1) \quad (1)$$

where $\nu_i = 1, \dots, q$. For each configuration $\{\nu_i\}$ associate to each site a particle of species r if $\nu_i = r$. The species may be characterised by q different colours. We consider clusters made of particles of species 1 connected by nearest-neighbour random bonds, each one having a probability p_B of being active and $1 - p_B$ of being non-active. In the following, unless explicitly mentioned, we shall always refer to particles and clusters of species 1.

To study this problem we want to give first a Hamiltonian formalism for such percolation problems and then apply the Migdal–Kadanoff renormalisation group (MKRG) (Migdal 1976, Kadanoff 1976). Note that from the Potts Hamiltonian given in (1) it is only possible to evaluate the Potts free energy, from which all the ‘thermal’ properties can be derived, such as the order parameter, the susceptibility and so on, but not the ‘connectivity’ properties which pertain to the percolation problem, such as the percolation probability, the mean cluster size and so on. Therefore, we need to find another Hamiltonian from which it is possible to obtain the generating function for the SBPCP. Kastelyn and Fortuin (1969, 1972) have shown that the generating function for the random bond percolation problem is given by

$$\left. \frac{dF_{\text{Potts}}(s)}{ds} \right|_{s=1} \equiv \lim_{s \rightarrow 1} \frac{F_{\text{Potts}}(s)}{s-1}$$

where $F_{\text{Potts}}(s)$ is the free energy of the s -state Potts model ($F_{\text{Potts}}(1) = 0$). As a generalisation of this result we shall show in the appendix that the generating function of the SBPCP is given by $dF_{\text{DP}}(s)/ds|_{s=1}$ where $F_{\text{DP}}(s)$ is the free energy of the following dilute s -state Potts model

$$-\beta\mathcal{H}_{\text{DP}} = -\beta\mathcal{H}_P + J \sum_{\langle ij \rangle} (\delta_{\sigma_i \sigma_j} - 1) \delta_{\nu_i, 1} \delta_{\nu_j, 1} - h \sum_i (\delta_{\sigma_i, 1} - 1) \delta_{\nu_i, 1} \quad (2)$$

where $\sigma_i = 1, \dots, s$ is an s -state Potts variable, \mathcal{H}_P is the q -state Potts Hamiltonian (1), which determines the particle distribution, and $p_B = 1 - \exp(-J)$ is the probability of having a link between two nearest-neighbour particles of species 1. h is the ghost field for such a percolation problem. In the following we shall always consider the Hamiltonian (2) in the limit $s \rightarrow 1$.

For $q = 2$ the Hamiltonian (2) coincides with the more familiar lattice gas Potts model (Nienhuis *et al* 1979, Murata 1979, Coniglio and Klein 1980, Wu 1980, Kondor and Temesvari 1981). Hamiltonian (2) can easily be generalised to study percolation of all the other species of particles, with different interactions and chemical potentials. This corresponds to a site bond Potts correlated polychromatic percolation, which contain as a special case the random site polychromatic percolation studied by Zallen (1977) and Halley and Holcomb (1978) (see appendix for more details). A different polychromatic site correlated percolation has been introduced by Stanley (1979) and Stanley and Teixeira (1980) to describe the anomalous properties of supercooled water.

In order to understand the MKRG analysis better, which will be done later, we note first some properties of the Hamiltonian (2).

(i) In the limit $H \rightarrow -\infty$ all the variables $\nu_i = 1$, and Hamiltonian (2) becomes a pure s -state Potts model, describing therefore random bond percolation in the $s \rightarrow 1$ limit (Kastleyn and Fortuin 1969, 1972). This is consistent with the fact that the SBPCP for $H \rightarrow -\infty$ becomes a random bond percolation since all the sites are occupied by particles of species 1.

(ii) For $J = h = 0$ Hamiltonian (2) coincides with the q -state Potts Hamiltonian (1).

(iii) Finally, it is not difficult to show that the Hamiltonian (2) is equivalent to an asymmetric $(s + 1 - q)$ -state Potts model (see Coniglio and Klein 1980, Coniglio *et al* 1981 for $q = 2$) namely

$$\begin{aligned}
 -\beta \mathcal{H}_{DP} = & K \sum_{\langle ij \rangle} (\delta_{\mu_i \mu_j} - 1) + (J - K) \sum_{\langle ij \rangle} (\delta_{\mu_i \mu_j} - 1) \left(\sum_{r=1}^s \delta_{\mu_i r} \right) \left(\sum_{r=1}^s \delta_{\mu_j r} \right) \\
 & - (H + \ln s) \sum_i \left(\sum_{r=1}^s \delta_{\mu_i r} - 1 \right) - h \sum_i \left(\delta_{\mu_i 1} - \sum_{r=1}^s \delta_{\mu_i r} \right) \tag{3}
 \end{aligned}$$

where μ_i is a new Potts variable which assumes $s - 1 + q$ values. If we choose $J = K$ and $H = h = 0$ in the limit $s \rightarrow 1$ Hamiltonian (3) becomes a q -state Potts model.

This is true for any q and d ; therefore the special clusters made of particles of species 1 connected by active links with probability $p_B = 1 - e^{-K}$, which we call ‘Potts droplets’, are expected to exhibit the same Potts singularities at the Potts critical point $K = K_c, H = 0$. That is, we expect that for $H = 0$ the linear dimension of these Potts droplets diverges as $\xi \sim (K - K_c)^{-\nu}$ and the mean cluster size S diverges as $S \sim (K - K_c)^{-\gamma}$ where ν is the Potts correlation length exponent and γ the susceptibility Potts exponent. Also we expect first-order percolation transitions for $q > q_c$ where the Potts model exhibits first-order thermal transitions.

We now apply the MKRG (Migdal 1976, Kadanoff 1976) to Hamiltonian (2) using the same procedure adopted by Berker *et al* (1978). The recursion relations in the limit $s \rightarrow 1$, for dimension $d = 2$ and coordination number of the lattice c , are

$$x' = x'(x, y, w, f, g, t) = D/C \tag{4a}$$

$$y' = y'(y, w, z) = I/G \tag{4b}$$

$$w' = w'(y, w, z) = w^2 FI/AG \tag{4c}$$

$$z' = z(y, w, z) = AG^2/F^2 I \tag{4d}$$

$$f' = f'(x, y, w, f, g, t) = f^2 BD/x^4 AC \tag{4e}$$

$$g' = g'(x, y, w, f, g, t) = x^4 AC^2/B^2 D \tag{4f}$$

$$t' = t'(x, y, w, z, f, g, t) = x^2 ACE/BDF \tag{4g}$$

where $x \equiv e^J, y \equiv e^K, w \equiv e^{H/c}, f \equiv e^{h/c}; z, g, t$ are auxiliary variables which we need to solve the system of equations in closed form, and

$$A = y^4 + (q - 1)w^4$$

$$B = x^2 y^4 + (x^2 - 1)y^4 f^4 g^2 + (q - 1)x^4 w^4 t^2$$

$$C = y^4 + 2(x^2 - 1)y^4 f^4 g^4 + (q - 1)x^4 w^4 t^4$$

$$D = y^4 + (x^4 - 1)y^4 f^4 g^4 + (q - 1)x^4 w^4 t^4$$

$$E = y^2 + (x^2 - 1)y^2 f^4 g^2 t^2 + (y^2 + q - 2)x^2 w^4 z^2 t^2$$

$$F = y^2 + (y^2 + q - 2)w^4 z^2$$

$$G = 1 + (2y^2 + q - 3)w^4 z^4$$

$$I = 1 + (y^4 + q - 2)w^4 z^4.$$

Note that equations (4b)–(4d) are independent of the variable J and coincide with the recursion relations of the q -state Potts Hamiltonian (1).

The case $q = 2$ (Ising correlation) has been studied by Coniglio and Klein (1980). Here, for convenience, we give the main results for this case.

Corresponding to $z^* = g^* = t^* = 1$ and $h^* = 0$, there are four non-trivial fixed points for the other variables.

$$(i) \quad J = J_1^*, \quad K = 0, \quad H = -\infty.$$

This corresponds to random bond percolation.

$$(ii) \quad J = J_2^* = 0, \quad K = K_c, \quad H = 0.$$

This corresponds to the Ising fixed point with the Ising scaling exponents $y_K = 0.747$, $y_H = 1.879$, and no percolation.

$$(iii) \quad J = J_3^* = K_c, \quad K = K_c, \quad H = 0.$$

This corresponds to percolation which occurs at the Ising critical point with the special value $p_B = 1 - e^{-K}$. Besides the Ising scaling exponents y_K and y_H there is also a scaling power related to the variable J , $y_{J_3} = 0.494$, while $y_h = y_H = 1.879$.

$$(iv) \quad J = J_4^*, \quad K = K_c, \quad H = 0.$$

This describes percolation which occurs at the Ising critical point for all values of p_B such that

$$1 - e^{-K_c} < p_B \leq 1. \quad (5)$$

Here besides the Ising scaling powers y_K , y_H we have $y_h = 1.945$ while $y_{J_4} < 0$. J is an irrelevant variable[†].

The interplay of all these fixed points gives rise to the phase diagram of figure 1(a). The percolation lines have been calculated for fixed values of p_B . All these lines end at the Ising critical point for values of p_B satisfying (5). For $p_B < 1 - e^{-K_c}$ they end above K_c . Any point, in such a phase diagram, where a percolation transition occurs, except the Ising critical point $H = 0$, $K = K_c$, is characterised by random percolation exponents, whereas if we approach the Ising critical point along the direction $H = 0$, $K \rightarrow K_c$, we find that the connectedness length ξ_p and the mean cluster size S diverge as

$$\xi_p \sim (K - K_c)^{-\nu_1} \quad S \sim (K - K_c)^{-\gamma_p}$$

for values of p_B given by (5). Here $\nu_1 = 1/y_K = 1$ is the two-dimensional Ising correlation length exponent, and $\gamma_p = 2(y_h - 1)/y_K = 1.890$, where the exact value $y_K = 1$ has been used. Note that this exponent is larger than the Ising susceptibility exponent $\gamma_I = 1.75$. This exponent was first calculated by Sykes and Gaunt (1976) who found,

[†] Note that in Coniglio and Klein (1980) there are two misprints relative to the values of y_{J_3} and y_h . The correct values are given here.

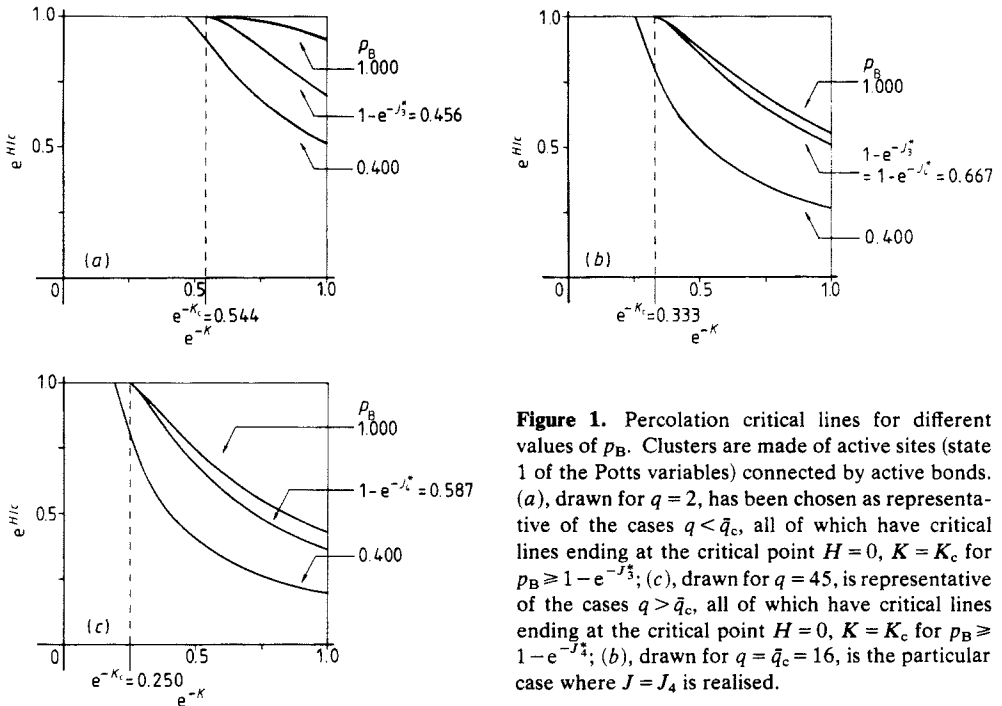


Figure 1. Percolation critical lines for different values of p_B . Clusters are made of active sites (state 1 of the Potts variables) connected by active bonds. (a), drawn for $q = 2$, has been chosen as representative of the cases $q < \bar{q}_c$, all of which have critical lines ending at the critical point $H = 0, K = K_c$ for $p_B \geq 1 - e^{-J_3^*}$; (c), drawn for $q = 45$, is representative of the cases $q > \bar{q}_c$, all of which have critical lines ending at the critical point $H = 0, K = K_c$ for $p_B \geq 1 - e^{-J_3^*}$; (b), drawn for $q = \bar{q}_c = 16$, is the particular case where $J = J_4$ is realised.

using series expansions, $\gamma_p = 1.91 \pm 0.01$. On the other hand, if we choose $p_B = 1 - e^{-K}$ we obtain

$$\xi_p \sim (K - K_c)^{-\nu_1} \quad S \sim (K - K_c)^{-\gamma_1}$$

both connectedness length and mean cluster size diverge with Ising exponents. These have been called Ising droplets since they diverge as the droplets in Fisher's droplet model (Fisher 1967). Note that p_B has been chosen as a function of temperature.

Of course one would have obtained the same critical behaviour even for a choice of $p_B = 1 - e^{-K_c}$ independent of the temperature, but with the choice $p_B = 1 - e^{-K}$ one can describe the droplets for all temperatures, due to the property of Hamiltonian (2) for $q = 2$ of being an Ising Hamiltonian for such a choice of p_B .

To complete the picture, we have drawn in figure 2(a) the critical threshold for the bond probability p_B as a function of H for fixed values of T . For $T = \infty$ we have site bond random percolation (Agrawal *et al* 1979). For $T = T_c$ there is a discontinuity at $H = 0$. The line at $H = 0$ corresponds to the bond percolation threshold on a dilute lattice which is made of the incipient infinite cluster of sites at the Ising critical point. As the point $p_B^* = 1 - e^{-K_c}$ is approached from small values of p_B , the connectedness length ξ_p and the mean cluster size S diverge as

$$\xi_p \sim (p - p_B^*)^{-\nu_B} \quad S \sim (p - p_B^*)^{-\gamma_B}$$

where $\nu_B = 1/y_{J_3} = 2.022$, $\gamma_B = 2(y_H - 1)/y_{J_3} = 3.539$, and the exact value $y_H = 1.875$ has been used. If p_B increases from p_B^* to 1 there is no formation of an infinite cluster with particle density larger than zero; instead we shall always be at the onset of an infinite cluster, with the incipient infinite cluster becoming more compact.

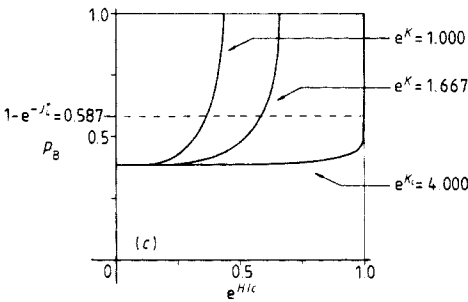
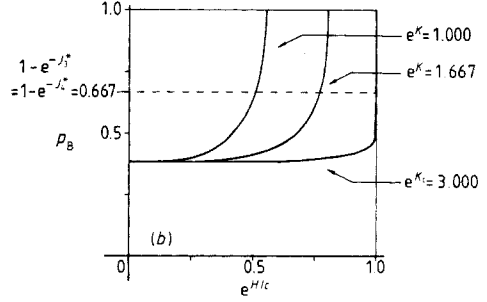
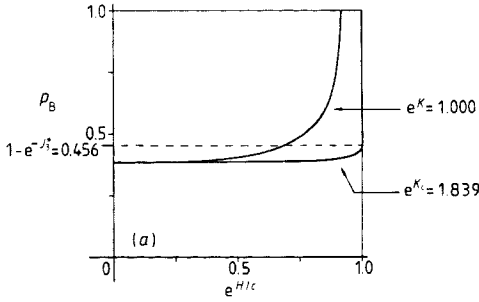


Figure 2. Phase diagram for different values of the temperature T : for a given choice of K (depending on T via the inserted Boltzmann factor β) infinite clusters of active sites connected by active bonds appear only above the related curve. (a), drawn for $q = 2$, is representative of all cases $q < \bar{q}_c$, for which the curve corresponding to T_c ends at the point $H = 0, p_B = 1 - e^{-J_3^*}$; (c), drawn for $q = 45$, is representative of all cases $q > \bar{q}_c$, for which the curve corresponding to T_c ends at the point $H = 0, p_B = 1 - e^{-J_4^*}$; in (b), drawn for $q = \bar{q}_c = 16$, we have $J_3^* = J_4^*$. Note the non-analytic behaviour of these percolation lines for $K = K_c$ and $H = 0$ in contrast, for example, to the more familiar site bond random case ($K = 0$). In (a), for major clarity, we have not drawn the curve at intermediate temperature $K = 5/3$ because it is very close to the curve at critical temperature $K = K_c$.

Now let us consider the case $q > 2$. We still found as before four non-trivial fixed points (see table 1): J_1^* , which corresponds to pure random bond percolation, and the other three which occur at the Potts critical point $H = 0, K = K_c$. $J_2^* = 0$ corresponds to the thermal Potts phase transition and the absence of percolation. $J_3^* = K_c$ is a fixed point where the scaling power relative to percolation quantities is the same as the thermal Potts scaling power. As for $q = 2$, this fixed point describes those special clusters, called droplets, which have the same singularities as the Potts model at its critical point. In particular for $q > q_c$ we expect a first-order percolation transition as for the thermal Potts transition, although the MKRG used here does not show any first-order transition except at $q = \infty$. The last fixed point occurs at $J = J_4^*$ where, as for $q = 2$, the scaling powers associated to K and H coincide with the Potts thermal scaling powers y_K and y_H but $y_h \neq y_H$. Therefore the connectedness length diverges as the thermal Potts correlation length, but the mean cluster size diverges with an exponent different from the susceptibility exponent.

An interesting new result is found for a particular value of $q = \bar{q}_c = 16$. Here the two fixed point lines $J_3^*(q)$ and $J_4^*(q)$ intersect (figure 3). For $q < \bar{q}_c$ the fixed point structure is the same as for $q = 2$ with $y_{J_3} > 0$ and $y_{J_4} < 0$, while for $q > \bar{q}_c, y_{J_3} < 0$ and $y_{J_4} > 0$.

For $q = \bar{q}_c, y_{J_3} = y_{J_4} = 0$: J is marginal!

We conjecture that \bar{q}_c coincides with the value q_c above which the thermal phase transition becomes of first order. The reason for such conjecture is better understood by looking at the result of the MKRG for $q = \infty$. For such values of $q = \infty$ we find at

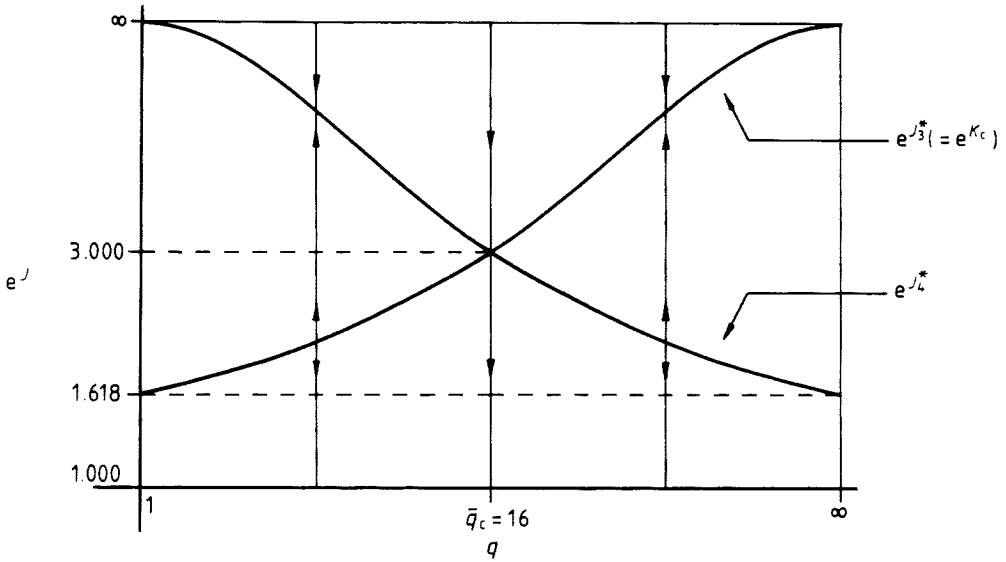


Figure 3. Schematic representation of the J_3^* and J_4^* dependence on q . Flow lines are drawn that show the interchange between repulsive ($y_J > 0$) and attractive ($y_J < 0$) behaviour of the two fixed points below and above \bar{q}_c ; at $q = \bar{q}_c = 16$ we have $J_3^* = J_4^*$ with $y_J = 0$. Note that in the limits $q \rightarrow 1$ and $q \rightarrow \infty$ the fixed points J_3^* and J_4^* respectively describe pure random bond percolation.

the fixed point $J_2^* = 0$, $K_c = \infty$, $H = 0$ the scaling power $y_K = y_H = 2$ describing a first-order transition in the thermal problem. At the fixed point describing the Potts droplets $J_3^* = K_c = \infty$ we have $y_K = y_H = 2$ with $y_{J_3} < 0$. This fixed point therefore also describes a first-order percolation transition of the droplets as $K \rightarrow \infty$ for all values of p_B such that $1 \geq p_B > 1 - e^{-J_4^*}$.

At the other fixed point we find $J_4^* = J_1^*$ and $y_{J_4} = y_{J_1}$ describing random bond percolation in the variable $p_B = 1 - e^{-J}$. This is easily understood because when $q = \infty$ the density of particles is unity at $K_c = \infty$ and then jumps to $1/q = 0$ for $K < K_c$.

Therefore at K_c the fixed point J_4^* describes random bond percolation in a lattice where all the sites are occupied. Since the fixed point structure is the same for all $q > \bar{q}_c$ we believe that the same first-order transition picture as for $q = \infty$ is also valid for these values of q . Therefore we are led to conjecture that $\bar{q}_c = q_c$, although the MKRG shows only first-order transitions for $q = \infty$.

In conclusion we have considered a q -state Potts model. Different species of particles or colours have been associated to the q different values of the Potts variables. We have presented a Hamiltonian formalism which enables us to study a percolation problem in which clusters are made of particles of species 1 connected by nearest-neighbour bonds, each one having a probability p_B of being active. For $d = 2$ we have applied the MKRG and found for $p_B = 1$ in the $H - T$ plane a line of percolation points which ends at the Potts critical point $H = 0$, $K = K_c$. At this point, due to the symmetry, the other species of particles also percolate at the same time. The clusters diverge with random percolation exponents along the curve, while at $H = 0$, $K = K_c$ the linear dimension of clusters diverges with the thermal Potts correlation length and the mean cluster size with an exponent different from the susceptibility exponent. If $p_B = 1 - e^{-K}$, where K is the Potts coupling constant, both the linear dimension and the mean

cluster size diverge with Potts exponents as $H = 0$ and $K \rightarrow K_c$. We have called these clusters Potts droplets. Above q_c , therefore, both types of clusters ($p_B = 1, p_B = 1 - e^{-K}$) exhibit first-order transitions. We have also found a value of $q = \bar{q}_c$ where p_B become a marginal variable. It has been conjectured that $\bar{q}_c = q_c$.

It would be of much interest to study this problem and to verify this conjecture by using other approaches.

Table 1. Fixed point values and scaling powers associated with the phase transitions in the Hamiltonian model (2) for some representative values of q .

q	e^{K^*}	γ_K	γ_H	e^{J^*}	γ_J	γ_h	Fixed point
2	1.839	0.747	1.879	1.839	0.495	1.879	J_3^*
				10.214	-1.487	1.945	J_4^*
16	3.000	1.170	1.835	3.000	0	1.835	$J_3^* = J_4^*$
				4.000	-0.322	1.834	J_3^*
45	4.000	1.356	1.834	2.419	0.226	1.818	J_4^*
				∞	$-\infty$	2.000	J_3^*
∞	∞	2.000	2.000	1.618	0.611	1.899	J_4^*

Appendix

Consider the Potts Hamiltonian on a regular lattice of N sites

$$-\beta \mathcal{H}_P = K \sum_{\langle ij \rangle} (\delta_{\nu_i \nu_j} - 1) - H \sum_i (\delta_{\nu_i 1} - 1) \tag{A1}$$

where each value of the variable $\nu_i = 1, \dots, q$ is associated to a particular species of particles. In the site bond Potts correlated percolation the clusters are made of particles of species 1 connected by nearest-neighbour random bonds, each one having probability p_B of being active. In the following, unless explicitly mentioned, we shall always refer to particles and clusters of species 1.

The quantities of interest in percolation are the average number of clusters of s particles per site $\langle n_s \rangle$, the average number of clusters per site

$$\langle N_{cl} \rangle = \sum s \langle n_s \rangle, \tag{A2}$$

the percolation probability

$$P = 1 - \sum' s \langle n_s \rangle \rho^{-1} \tag{A3}$$

(ρ is the density of particles), the mean cluster size

$$S = \sum' s^2 \langle n_s \rangle \left(\sum' s \langle n_s \rangle \right)^{-1} \tag{A4}$$

and the pair connectedness function

$$P_{ij} = \langle \gamma_{ij} \rangle \tag{A5}$$

where γ_{ij} is 1 if the sites i and j belong to the same finite cluster, 0 otherwise. Σ' is the sum over all finite clusters. Here the brackets stand for the average over the Potts

and bond configurations, namely

$$\langle \dots \rangle = \lim_{N \rightarrow \infty} \sum_{\{\nu_i\}} \exp(-\beta \mathcal{H}_P) \left(\sum_{C \subseteq E\{\nu_i\}} \dots p_B^{|C|} (1 - p_B)^{|D|} \right) \left(\sum_{\{\nu_i\}} \exp(-\beta \mathcal{H}_P) \right)^{-1} \quad (A6)$$

where $E\{\nu_i\}$ is the set of all bonds in the sublattice made of sites occupied by the particles of species 1 in the configuration ν_i . C is a subset of $E\{\nu_i\}$ and $D = E\{\nu_i\} - C$; $|C|$ and $|D|$ are the number of bonds respectively in the subsets C and D . Note that

$$\sum_{C \subseteq E\{\nu_i\}} p_B^{|C|} (1 - p_B)^{|D|} = 1$$

for every configuration $\{\nu_i\}$.

We shall show now that we can obtain equations (A2)–(A5) from the free energy of the following dilute q -state Potts model

$$-\beta \mathcal{H}_{DP} = -\beta \mathcal{H}_P + J \sum_{\langle ij \rangle} (\delta_{\sigma_i \sigma_j} - 1) \delta_{\nu_{i1}} \delta_{\nu_{j1}} - h \sum_i (\delta_{\sigma_i 1} - 1) \delta_{\nu_{i1}} \quad (A7)$$

where $\sigma_i = 1, \dots, s$ are also Potts variables and \mathcal{H}_P is given by (A1). We give the derivation here, following closely the procedure of Murata (1979) and Coniglio and Klein (1980) (see also Coniglio *et al* (1981) for more details).

Starting from (A6) we can write the partition function in the following way:

$$Z_{DP} = \sum_{\{\nu_i\}} \exp(-\beta \mathcal{H}_P) s^{\sum_i (1 - \delta_{\nu_{i1}})} Z_{\text{Potts}}\{\nu_i\} \quad (A8)$$

where

$$Z_{\text{Potts}}\{\nu_i\} = \sum_{\{\sigma_i\}} \prod_{\langle ij \rangle \in E\{\nu_i\}} (q_B + p_B \delta_{\sigma_i \sigma_j}) \prod_{i \in \theta\{\nu_i\}} [\exp(h) + (1 - \exp(h)) \delta_{\sigma_i 1}]$$

is the partition function of the s -state Potts model defined on the sublattice made of those sites occupied by particles of species 1 in the configuration $\{\nu_i\}$. $\theta\{\nu_i\}$ is the set of vertices in this sublattice and $E\{\nu_i\}$ the set of bonds; $q_B = \exp(-J)$, $p_B = 1 - q_B$. In (A8) the term $s^{\sum_i (1 - \delta_{\nu_{i1}})}$ is due to the trace over all the sites occupied by particles of species different from 1. Following the usual procedure as for the regular lattice (Wu 1978), (A9) becomes

$$Z_{\text{Potts}}\{\nu_i\} = \sum_{C \subseteq E\{\nu_i\}} p_B^{|C|} q_B^{|D|} \prod_r [(s - 1) \exp(hs_r) + 1] \quad (A10)$$

where r labels the clusters in the configuration C and s_r is the number of sites in the r th cluster. Equations (9) and (10) give the same distribution as in (A6).

Let us now define the generating function G in the following way:

$$G = \lim_{N \rightarrow \infty} \frac{1}{N} \left(\frac{d}{ds} \ln Z_{DP} \right)_{s=1} \quad (A11)$$

From (A10) we have

$$G(h = 0) = 1 - \rho + \langle N_{cl} \rangle \quad (A12)$$

$$\left(\frac{dG}{dh} \right)_{h=0} = \sum' s \langle n_s \rangle \quad (A13)$$

$$\left(\frac{d^2 G}{dh^2} \right)_{h=0} = \sum' s^2 \langle n_s \rangle \quad (A14)$$

which are related to equations (A2)–(A4). Analogously the pair connectedness function p_{ij} can be obtained by introducing inhomogeneous fields h_i at each site i :

$$p_{ij} = \left(\frac{\partial^2 G}{\partial h_i \partial h_j} \right)_{h_i=h_j=0}. \quad (\text{A15})$$

It is convenient to note that with the formalism described here one can treat a very general percolation problem, in which the sites may be occupied by different species of particles, each particle interacting only with particles of the same species r with a nearest-neighbour coupling constant K_r and with a chemical potential H_r . The clusters are made with particles of the same species connected by nearest-neighbour links, each one having a probability $p_{Br} = 1 - \exp(-J_r)$ of being active. This general percolation problem is obtained by generalising (A1) and (A2) in the following way:

$$-\beta \mathcal{H}_P - \sum_{\langle ij \rangle} \sum_{r=1}^q K_r (\delta_{\nu_{ir}} \delta_{\nu_{jr}} - 1) - \sum_i \sum_{r=1}^q H_r (\delta_{\nu_{ir}} - 1) \quad (\text{A16})$$

$$-\beta \mathcal{H}_{DP} = -\beta \mathcal{H}_P + \sum_{\langle ij \rangle} \sum_{r=1}^q J_r (\delta_{\sigma_i \sigma_j} - 1) \delta_{\nu_{ir}} \delta_{\nu_{jr}} - \sum_{r=1}^q h_r (\delta_{\sigma_{i1}} - 1) \delta_{\nu_{ir}} \quad (\text{A17})$$

h_r is the ghost field relative to the particles of species r .

As before we can define the generating function for the species r , $G_r = (d F_{DP}^{(r)} / ds)_{s=1}$ where $F_{DP}^{(r)}$ is the free energy per site obtained from (A6) in which $J_t = 0$ for any t except $t = r$. The derivatives of G_r with respect to h_r give all the other percolation quantities relative to species r .

This general percolation problem, which we call site bond Potts correlated polychromatic percolation, contains as a particular case the random site polychromatic percolation treated by Zallen (1977) and Halley and Holcomb (1978) if one chooses $p_r = 1$ and $K_r = 0$ for any species r .

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